Tropical Hyperfield and Modifications

Motivating problems:

① Define a tropical hypersurface as a "zero set"

② Study non-transverse intersections

③ Define a tropical version of birational equivalence

(Section to develop)

(1) Let X be a tropical hypersurface defined by a tropical polynomial f.

Since $-\infty$ is the zero element of the tropical semi-field $\mathbb{T}$, one would want to define X as

$$\{x : f(x) = -\infty\}.$$

Does not make sense with the tropical <u>semi-field</u>.

"Solution": replace $(\mathbb{T}, \oplus, \otimes)$ by the <u>tropical hyperfield</u> $(\mathbb{T}, \boxplus, \boxtimes)$.

- $\boxtimes$ is again the addition
- $\boxplus$ is defined as follows:

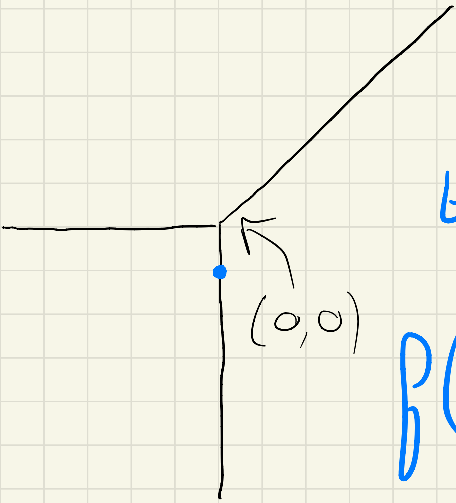
for $a, b \in \mathbb{T}$, we have

$$a \boxplus b = \begin{cases} \max(a,b) & \text{if } a \neq b \\ \{x \in \mathbb{T} \mid x \leq a\} & \text{if } a = b \end{cases}$$

↳ We allow the 1st operation of a hyperfield to be multivalued.

$\underline{\text{Ex}}$: $f(x,y) = x \boxplus y \boxplus 0$

The point $(0,-1)$ lie on the associated tropical line.



$$f(0,-1) = 0 \boxplus (-1) \boxplus 0$$
$$= \{x \in \mathbb{T} \mid x \leq 0\}$$

Then $-\infty \in f(0,-1)$

For f defining a tropical hypersurface X,

$$\underline{x} \in X \iff -\infty \in f(\underline{x}).$$

Note: Since we still do not work over a field, we obtain only

$$-\infty \in f(\underline{x}) \text{ and not}$$

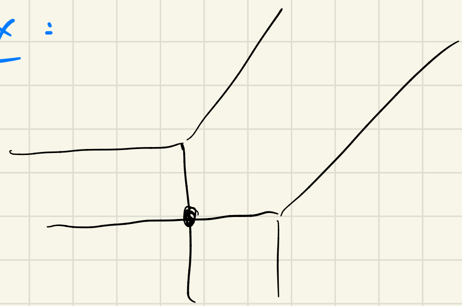
$$-\infty = f(\underline{x}).$$

② Given n tropical hypersurfaces in $\mathbb{T}^n$ of degree $d_1, \dots, d_n$, one would want to obtain

$$d_1 \cdot \ldots \cdot d_n \quad \text{intersection points}$$

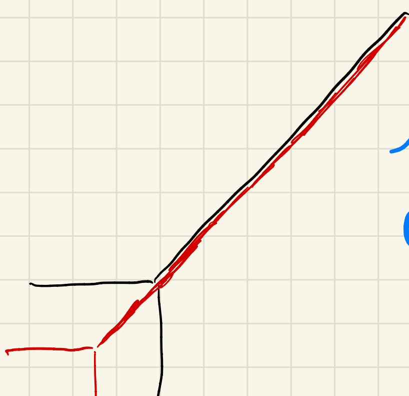
(counted with multiplicity).

Ex:



Two tropical lines in $\mathbb{R}^2$ intersecting transversely

$\hookrightarrow$ OK



Here, we obtain a 1-dimensional intersection component.

1st solution: up to small translation, we obtain a transverse intersection, and we can count the number of intersection points there.

Remaining problem: Where does the intersection points $\mathrm{Trop}(\mathcal{X}_1 \cap \dots \cap \mathcal{X}_n)$ lie in

$$\mathrm{Trop}(\mathcal{X}_1) \cap \dots \cap \mathrm{Trop}(\mathcal{X}_n) ?$$

Def: Let X be a tropical hypersurface in $\mathbb{T}^n$, being the "zero set" of a tropical polynomial f.

The modification of $\mathbb{T}^n$ along X is the set

$$m_X(\mathbb{T}^n) = \{ (\underline{x}, y) \in \mathbb{T}^n \times \mathbb{T} \mid y \in f(\underline{x}) \}.$$

Prop: The set $m_X(\mathbb{T}^n)$ coincides with the zero set of the tropical polynomial

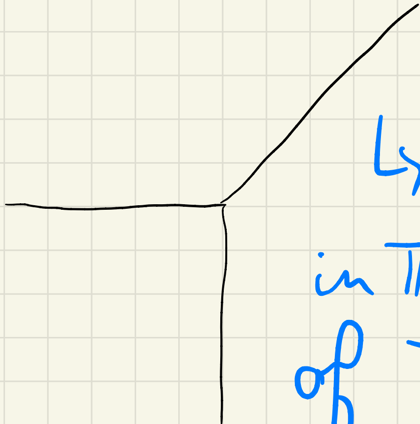
$$f'(\underline{x}, y) = f(x) \boxplus y$$

$\underline{\text{Ex}}$:

$$f(x) = x \boxplus 0$$

$$f'(x,y) = f(x) \boxplus y$$

$$= x \boxplus y \boxplus 0$$



↳ The tropical line in $\mathbb{T}^2$ is the modification of $\mathbb{T}$ along the zero set of f.

For $Y \subset \mathbb{T}^n$ a tropical variety, and $Y' \subset \mathfrak{m}_X(\mathbb{T}^n)$ a tropical variety, Y' is a modification of Y along X

if the natural projection

$$p : \mathbb{T}^n \times \mathbb{T} \longrightarrow \mathbb{T}^n$$

(to define)

restricted to Y' is a tropical morphism $p|_{Y'} : Y' \longrightarrow Y$

of degree one.

In that case, we write

$$Y' = \mathfrak{m}_X(Y).$$

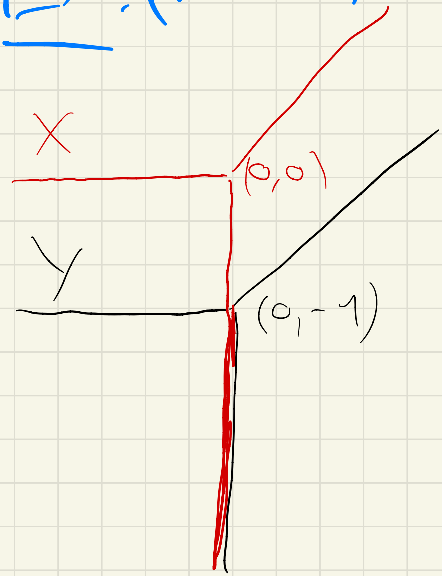
Note: In the previous definition, the choice of X and Y does not determine uniquely a tropical modification Y'.

If $X = \mathrm{Trop}(\mathcal{X})$ and $Y = \mathrm{Trop}(\mathcal{Y})$ tropical hypersurfaces, with $\mathcal{X} = V(f)$ and $\mathcal{Y} = V(g)$, we can define Y' "uniquely" using higher order terms in f and g.

Ex: (B-LdM)

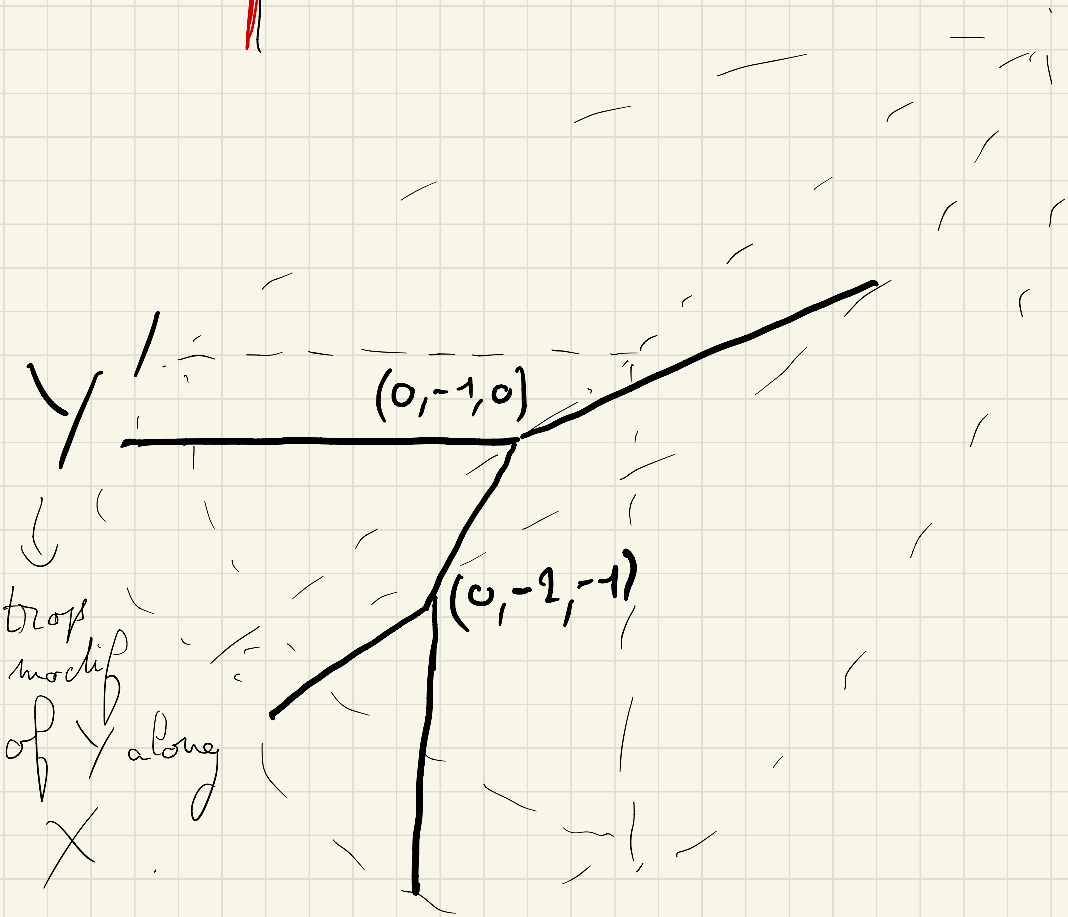
$$\mathcal{X} = V\big((1+t^2)+x+y\big)$$

$$\mathcal{Y} = V\big((1+t)+x+t^{-1}y\big)$$



$$X = \mathrm{Trop}(\mathcal{X})$$

$$Y = \mathrm{Trop}(\mathcal{Y})$$



Then $\operatorname{Trop}(\mathcal{X} \cap \mathcal{Y})$

$$= Y' \cap \{z = -\infty\}$$

$$= (0, -2)$$

↳ We can easily compute that the lines $\mathcal{X}$ and $\mathcal{Y}$ intersect in $p = (-1, -t^2) \in (\mathbb{C}\{\{t\}\}^\times)^2$ and $\operatorname{Trop}(p) = (0, -2) \in \mathbb{T}^2$

✓

General process:

P polynomial in $\mathbb{C}\{\{t\}\}[x_1, \dots, x_n]$

$\mathcal{X} = V(P)$ $\qquad$ $\overset{\shortparallel}{\mathbb{K}}[x_1, \dots, x_n]$

$$\mathcal{X}' = (\mathbb{K}^\times)^n \setminus \mathcal{X}$$

$$\hookrightarrow \ \phi : \mathcal{X}' \hookrightarrow (\mathbb{K}^\times)^{n+1}$$

$$\underline{x} \longmapsto (\underline{x}, P(\underline{x}))$$

$X' = \mathrm{Trop}(\phi(\mathcal{X}'))$ is the tropical modification of $\mathbb{R}^n$ defined by P.

For $\mathcal{Y} \subset (\mathbb{K}^{\times})^n$ hypersurface with $Y = \mathrm{Trop}(\mathcal{Y})$, the tropical modification of Y along X, with respect to P, is $Y' = \mathrm{Trop}(\phi(\mathcal{Y} \cap \mathcal{X}'))$.

Prop (TO CHECK):

$$\mathrm{Trop}(\mathcal{X} \cap \mathcal{Y}) = Y' \cap \{x_{n+1} = -\infty\}$$

with

$$Y' = \mathrm{Trop}(\phi(\mathcal{Y} \cap \mathcal{X}')).$$

③

Def: We say that two tropical varieties X and Y are equivalent if they are related by a chain of tropical modifications and reverse operations

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